American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2009-2010

quiz # 2 - solution

1. for each of the following functions, find the domain and the range.

a) $f(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$

$D_f = \{(x,y,z) \in \mathbf{R}^3 / x^2 + y^2 + z^2 \leq 9\}$; Range: $[0,3]$; boundary: $\{x^2 + y^2 + z^2 = 9\}$; the domain is closed cause it contains its boundary, and bounded.

b) $g(x,y) = \frac{x}{x^2 - y}$

$D_g = \{(x,y) \in \mathbf{R}^2 / y \neq x^2\}$; Range: $\mathbf{R}$; boundary: $\{y = x^2\}$; the domain is open cause it does not contain its boundary, and unbounded.

2. *remark:* take to the same denominator, then use series representation of cos

$$\lim_{t\to 0}\left(\frac{1}{2-2\cos t} - \frac{1}{t^2}\right) = \lim_{t\to 0}\left(\frac{t^2 - 2 + 2\cos t}{t^2(2 - 2\cos t)}\right) = \lim_{t\to 0}\left(\frac{t^2 - 2 + 2(1 - \frac{t^2}{2} + \frac{t^4}{24} + o(t^6))}{t^2(2 - 2(1 - \frac{t^2}{2} + \frac{t^4}{24} + o(t^6)))}\right)$$

$$= \lim_{t\to 0}\left(\frac{\frac{t^4}{12} + o(t^6)}{t^4 - \frac{t^6}{12} + o(t^6)}\right) = \lim_{t\to 0}\left(\frac{\frac{1}{12} + o(t^2)}{1 - \frac{t^2}{12} + o(t^4)}\right) = \frac{1}{12}$$

3. give the Taylor series expansion of $f(x) = \frac{2+x}{(1-x)(1+2x)}$ at $x = -1$, then find $f^{(n)}(-1)$

remark: do not derive f directly! use a substitution

Let $u = x+1$, then $g(u) = f(u-1) = \dfrac{u+1}{(2-u)(2u-1)} = \dfrac{1}{2-u} + \dfrac{1}{2u-1} = \dfrac{1}{2}.\dfrac{1}{1-\frac{u}{2}} - \dfrac{1}{1-2u}$

The Taylor series of f at $x = -1$ is the Maclaurin series of g, hence

$$g(u) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n - \sum_{n=0}^{\infty}(2u)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 2^n\right)u^n \text{ , and}$$

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 2^n\right)(x+1)^n \text{ is the Taylor series of } f \text{ at } x = -1$$

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n \text{ , then by comparison, } f^{(n)}(-1) = n! \times \left(\frac{1}{2^{n+1}} - 2^n\right)$$

method 2 : write f in fractions then derive !

$$f(x) = \frac{2+x}{(1-x)(1+2x)} = \frac{1}{1-x} + \frac{1}{1+2x}$$

$$f'(x) = \frac{1}{(1-x)^2} - \frac{2}{(1+2x)^2}$$

$f''(x) = \dfrac{2}{(1-x)^3} + \dfrac{4}{(1+2x)^3}$, and hence $f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}} + \dfrac{(-1)^n n! 2^n}{(1+2x)^{n+1}}$, and then $f^{(n)}(-1) = \dfrac{n!}{2^{n+1}} - n!2^n$

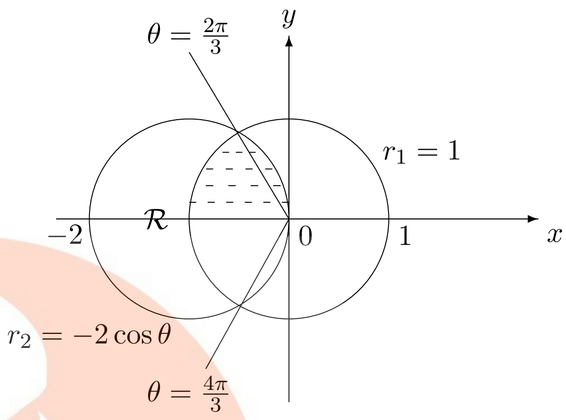
4. find the area inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$

<u>*remark:*</u> sketch the curves; find the points of intersection, then find the area.

<u>*method 1*</u> : Area$= \displaystyle\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}((-2\cos\theta)^2 - 1)d\theta = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ (by using the rule $cos(2\theta) = 2\cos^2\theta - 1$)

the area can also be calculated by another way

<u>*method 2*</u> : (longer!)



Area($\mathcal{R}$)$= \pi - 2\times$Area(shaded region)

$$= \pi - 2 \times \left(\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}(-2\cos\theta)^2 d\theta + \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2} d\theta \right)$$

$$= \pi - 2 \times \left(\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (\cos(2\theta) + 1) d\theta + \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2} d\theta \right)$$

$$= \pi - 2 \times \left[\tfrac{\sin(2\theta)}{2} + \theta \right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} + \tfrac{\pi}{3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}$$

5. a. $\displaystyle\left|\frac{x^2y}{2x^2+y^2}\right| \le \left|\frac{x^2y}{2x^2}\right| \le \frac{|y|}{2} \longrightarrow 0$ as $(x, y) \to (0, 0)$,

then by sandwich theorem, $f(x, y) \longrightarrow 0$ as $(x, y) \to (0, 0)$

b. consider the path $y = m(x-1)$; note that $(1, 0)$ belongs to this path!

$\displaystyle f(x, m(x-1)) = \frac{m(x-1)^2}{x^2 + m^2(x-1)^2} = \frac{m}{1+m^2} \longrightarrow \frac{m}{1+m^2}$ as $(x, y) \to (1, 0)$;

the limit depends on m, then by the two path test, f has no limit at $(1, 0)$

6. a. $\displaystyle a_0 = \frac{1}{2\pi}\left(\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} dx\right) = \frac{\pi}{4} + \frac{1}{2}$

b. $\displaystyle f(x) = \tfrac{\pi}{4} + \tfrac{1}{2} + \sum_{n=1}^{+\infty} \frac{1}{\pi}\,\frac{(-1)^n - 1}{n^2}\cos(nx) + \sum_{n=1}^{+\infty} b_n \sin(nx)$

$x = \pi$ is a point of discontinuity of f; then at $x = \pi$ the series converges to $\frac{f(\pi^+)+f(\pi^-)}{2}$

$$\tfrac{\pi}{2} + \tfrac{1}{2} = \tfrac{\pi}{4} + \tfrac{1}{2} + \sum_{n=1}^{+\infty} \frac{1}{\pi}\,\frac{(-1)^n - 1}{n^2}\cos(n\pi)$$

split the sum into even and odd gives

$$\tfrac{\pi}{4} = \sum_{n=1}^{+\infty} \frac{1}{\pi}.\frac{(-1)^n - 1}{n^2}.(-1)^n = \sum_{n=0}^{+\infty} \frac{1}{\pi}.\frac{(-1)^{2n+1} - 1}{(2n+1)^2}.(-1)^{2n+1} + \sum_{n=1}^{+\infty} \frac{1}{\pi}.\frac{(-1)^{2n} - 1}{(2n)^2}.(-1)^{2n}$$

the second term is equal to 0, then $\displaystyle\tfrac{\pi}{4} = \sum_{n=0}^{+\infty} \frac{1}{\pi}.\frac{2}{(2n+1)^2}$

and finally $\displaystyle\tfrac{\pi^2}{8} = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$